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More About the Massive Multi-flavor Schwinger Model†

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The massive multi-flavor Schwinger model on a circle is reduced to a finite dimensional quantum mechanics problem. The model sensitively depends on the parameter $mL|\cos\frac{1}{2}\theta|$ where m, L, and θ are a typical fermion mass, the volume, and the vacuum angle, respectively.

The Schwinger model is QED in two dimensions. Its Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \sum_{a=1}^{N} \overline{\psi}_a \Big\{ \gamma^\mu(i\partial_\mu - eA_\mu) - m_a \Big\} \psi_a \ . \tag{1}$$

Schwinger showed [1] in 1962 that for $N = 1$ and $m = 0$, the gauge boson acquires a mass μ given by $\mu^2 = e^2/\pi$. The original massless Schwinger model is exactly solvable. It admits the chiral condensate $\langle \overline{\psi}\psi \rangle \neq 0$, the θ vacuum, and the confinement phenomenon. With a non-vanishing fermion mass the model is not exactly solvable. Still, if $m \ll \mu$, the model can be approximately solved.

When the number of flavor N is more than one, something puzzling happens. Coleman analyzed the $N = 2$ massive Schwinger model in 1976.[2] With the aid of the bosonization technique, he showed that there appear two bosons. The first one is essentially the same as the gauge boson in the $N = 1$ model. For $m_1 = m_2 \ll e$, its mass squared is doubled: $\mu_1^2 = 2e^2/\pi$. A surprising finding was the second boson picks up a mass given by

$$\mu_2 \sim m^{2/3} \mu_1^{1/3} \Big| \cos \tfrac{1}{2}\theta \Big|^{2/3} \qquad \text{(Coleman 76).} \tag{2}$$

The expression involves fractional powers of m and $|\cos\frac{1}{2}\theta|$. It is essentially "nonperturbative" in nature. This prompts us to ponder. Why can't we apply a perturbation theory in fermion masses?

If all fermion masses vanish, the model (1) is exactly solvable for arbitrary N. The $U(1)$ chiral symmetry is broken by anomaly. It is known that there result one massive boson

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$(\mu_1^2 = Ne^2/\pi)$ and $N-1$ massless bosons. $SU(N)$ chiral symmetry remains unbroken, but correlators show power-law decay behavior.

We analyze the model, placing it on a circle. There are new features which are absent on a line or in the Minkowski spacetime. First the constant mode of gauge field A_1 becomes a dynamical variable, which we call the Wilson line phase θ_W. It has been known that θ_W couples to fermion degrees of freedom through chiral anomaly, leading to the θ vacuum.[3] Secondly, we are going to show that constant modes (zero modes) of fermions along the circle play a decisive role in determining the structure of the vacuum when $N > 1$. It will be seen that the puzzle about (2) is also solved naturally on the way.

The result presented in this report is based on the work done in collaboration with Jim Hetrick and Satoshi Iso.[4]

The model carries several parameters. Let's take $m_a = m$ for the moment. Then the parameters are e, m, θ (vacuum angle), and L. In the Minkowski spacetime limit $L \to \infty$, there are only two dimensionless parameters: $\big(m/e, \theta\big)$. On a circle there appears one more dimensionless parameter, say, mL. We shall see that the theory sensitively depends on this new parameter. More precisely speaking,

$$\boxed{\;*** \; mL|\cos\tfrac{1}{2}\theta| \gg 1 \; *** \qquad \neq \qquad *** \; mL|\cos\tfrac{1}{2}\theta| \ll 1 \; ***\;} \tag{3}$$

This is a highly nontrivial statement. It implies that limits $L \to \infty$, $m \to 0$, and $\theta \to \pm\pi$ do not commute with each other. If one takes the Minkowski limit $L \to \infty$ with $m \neq 0$ and $|\theta| < \pi$, then one is always in the regime on the left side. However, if we take a large, but finite L and take the massless fermion limit $m \to 0$, then one will be in the regime on the right side. The massless limit in the Minkowski spacetime is singular, although it is smooth so long as L is kept finite.

To see these, we first adopt the bosonization on a circle. Suppose that fermion fields $\psi_a = (\psi_+^a, \psi_-^a)$ obey boundary conditions $\psi_a(t, x+L) = -\, e^{2\pi i\alpha_a}\, \psi_a(t,x)$. We bosonize in the interaction picture:

$$\begin{aligned}
&\psi_\pm^a(t,x) = \frac{1}{\sqrt{L}}\, C_\pm^a \, e^{\pm i\{q_\pm^a + 2\pi p_\pm^a (t\pm x)/L\}} \; : \, e^{\pm i\phi_\pm^a(t,x)} \; : \\
&C_+^a = \exp\Big\{ i\pi \sum_{b=1}^{a-1} (p_+^b + p_-^b - 2\alpha_b) \Big\} \quad , \quad C_-^a = \exp\Big\{ i\pi \sum_{b=1}^{a} (p_+^b - p_-^b) \Big\} \ .
\end{aligned} \tag{4}$$

Here bosonic variables satisfy

$$\begin{aligned}
&[q_\pm^a, p_\pm^b] = i\,\delta^{ab} \quad , \quad [c_{\pm,n}^a, c_{\pm,m}^{b,\dagger}] = \delta^{ab}\delta_{nm} \\
&\phi_\pm^a(t,x) = \sum_{n=1}^\infty \frac{1}{\sqrt{n}} \Big\{ c_{\pm,n}^a \, e^{-2\pi i n(t\pm x)/L} + \text{h.c.} \Big\} \ .
\end{aligned} \tag{5}$$

The boundary conditions are implemented by imposing the physical state condition

$$e^{2\pi i p_\pm^a} \,|\,\text{phys}\,\rangle = e^{2\pi i \alpha_a} \,|\,\text{phys}\,\rangle \quad . \tag{6}$$

The only physical gauge field degree of freedom is the Wilson line phase $e^{i\theta_W} = e^{ie\int_0^L dx\, A_1}$. The Hamiltonian written in the Schrödinger picture becomes

$$\begin{aligned} H &= H_0 \,+\, H_\chi \,+\, H_{\text{mass}} \\ H_0 &= -\frac{\pi\mu^2 L}{4}\frac{d^2}{d\theta_W^2} + \frac{\pi}{2L}\sum_{a=1}^N \Big\{ (p_+^a - p_-^a)^2 + (p_+^a + p_-^a + \frac{\theta_W}{\pi})^2 \Big\} \\ H_\chi &= \int_0^L dx\, \frac{1}{2}\Big\{ N_\mu[{\Pi_1}^2 + \chi_1'^2 + \mu^2\,\chi_1^2] + \sum_{\alpha=2}^N N_0[\Pi_\alpha^2 + \chi_\alpha'^2] \Big\} \end{aligned} \tag{7}$$

Here $\chi_1 = N^{-1/2}\sum_a (\phi_+^a + \phi_-^a)$ represents the $U(1)$ 'charge' part of oscillatory modes. $\chi_2 \sim \chi_N$ are other orthogonal combinations. Because of the Coulomb interaction χ_1 acquires a mass-squared $\mu^2 = Ne^2/\pi$. H_{mass} represents the fermion mass term.

It immediately follows from the bosonized form (7) that the massless fermion theory is trivial. It is a free theory. H_0 contains a non-trivial coupling resulting from the chiral anomaly, but is bilinear in dynamical variables so that it can be solved exactly. In particular, the oscillatory mode part H_χ consists of 1 massive boson and $N-1$ massless bosons. These $N-1$ massless bosons generate $SU(N)$ current algebra.

The presence of the mass term makes the theory nontrivial, and indeed very rich. Let's restrict ourselves to the $N=2$ (two flavor) case. Eigenstates of H_0 are

$$\begin{aligned} \Phi_s^{(n,r)} &= \frac{1}{(2\pi)^2}\, u_s(\theta_W + 2\pi n + \pi r + \pi\alpha_1 + \pi\alpha_2) e^{i(n+\alpha_1)(q_+^1 + q_-^1) + i(n+r+\alpha_2)(q_+^2 + q_-^2)} \\ E_s^{(n,r)} &= \ \mu s + \frac{\pi}{L}(r - \alpha_1 + \alpha_2)^2 + \text{const.} \end{aligned} \tag{8}$$

Here $u_s(x)$ is the s-th eigenfunction in the harmonic oscillator problem. The energy does not depend on n due to the invariance under large gauge transformations. Consequently it is more natural to take θ eigenstates $\Phi_s^r(\theta) = \sum_n e^{in\theta}\,\Phi_s^{(n,r)}$.

If the mass term H_{mass} is absent and $\alpha_1 = \alpha_2$, then $\Phi_0^0(\theta)$ is the lowest energy state, namely the vacuum. It follows that $\langle \overline{\psi}_a \psi_a \rangle_\theta = 0$.

Now let us look at effects of the mass term $H_{\text{mass}} = \int_0^L dx \sum_a m_a \overline{\psi}_a \psi_a$. Note $\psi_-^{a\dagger}\psi_+^a \propto e^{i(q_-^a + q_+^a)}\ N_0[e^{i\sqrt{2\pi}\chi_1}]\, N_0[e^{\pm i\sqrt{2\pi}\chi_2}]$. Hence, H_{mass} gives (1) transitions among various $\Phi_s^r(\theta)$'s, (2) a mass for χ_2, and (3) other interactions. Notice that we need to shift the reference mass of the normal product from $N_0[e^{i\alpha\chi_a(x)}]$ to $N_{\mu_a}[e^{i\alpha\chi_a(x)}]$ where μ_a is the physical mass of χ_a to be determined.

Having in mind that we are mostly interested in physics in a large volume, we, for the moment, suppress transitions to higher s states. Then the mass term gives transitions among $\Phi^{(n,r)}$'s:

$$\begin{aligned} \Phi^{(n,r)} &\stackrel{m_1}{\longrightarrow} \Phi^{(n\pm1,r\mp1)} \\ &\stackrel{m_2}{\longrightarrow} \Phi^{(n,r\pm1)} \end{aligned}$$

Our strategy is to determine all matrix elements of $H_{\rm mass}$ in the basis of $\Phi^{(n,r)}$, and diagonalize $H_0 + H_{\rm mass}$ exactly.

Indeed, this can be done without approximation. A close examination shows that the two mass parameters m_1 and m_2 enter in the combination of

$$\begin{gathered} m_1 e^{-i\theta} + m_2 ~=~ \overline{m}(\theta)~ e^{+i\bar\delta(\theta)} \\ \hbox{for}~~ m_1=m_2, \quad \overline{m} = 2m\left|\cos\tfrac{1}{2}\theta\right| \quad , \quad \bar\delta = -\tfrac{1}{2}\theta + \pi\Big[\frac{\theta+\pi}{2\pi}\Big] \end{gathered} \tag{9}$$

$\bar\delta(\theta)$ has a discontinuity at $\theta=\pm\pi$ where $\overline{m}(\theta)$ vanishes.

Exact eigenstates of the total Hamiltonian are sought in the form

$$\begin{aligned} |\Phi(\theta)\,\rangle &= \int_0^{2\pi} d\varphi~ f(\varphi+\bar\delta)\,|\Phi(\theta;\varphi)\,\rangle \\ \Phi(\theta,\phi) &= \frac{1}{2\pi}\sum_{n,r} e^{in\theta+ir\phi}\Phi^{(n,r)} \end{aligned} \tag{10}$$

The equation $(H_0+H_{\rm mass})|\Phi(\theta)\,\rangle = (\pi\epsilon/L)|\Phi(\theta)\,\rangle$ reads

$$\boxed{\bigg\{\Big(i\frac{d}{d\varphi} - \delta\alpha\Big)^2 - \kappa\,\cos\varphi\bigg\} f(\varphi) = \epsilon\, f(\varphi)} \tag{11}$$

where

$$\begin{aligned} \delta\alpha &= \alpha_1 - \alpha_2 \quad , \\ \kappa &= \frac{2}{\pi}\,\overline{m}L\,B(\mu_1 L)^{1/2}B(\mu_2 L)^{1/2}e^{-\pi/2\mu L} \quad , \\ B(\mu L) &= \begin{cases} 1 & \hbox{at}~~\mu L = 0 \\ \dfrac{\mu L e^\gamma}{4\pi} & \hbox{for}~\mu L \gg 1~. \end{cases} \end{aligned} \tag{12}$$

Eq. (11) is nothing but the equation describing a quantum pendulum ! The solution is controlled by the parameters κ and $\delta\alpha$. In a large volume $\kappa \gg 1$ so that the cosine potential term in (11) dominates. $f(\varphi)$ is localized around $\varphi=0$.

$$f(\varphi) \sim \exp\Big(-i\delta\alpha\varphi - \sqrt{\frac{\kappa}{8}}~\varphi^2\Big) \qquad \hbox{for}~~\kappa\gg 1~. \tag{13}$$

On the other hand, in a small volume limit $\kappa \ll 1$. $f(\varphi)$ shows sensitive dependence on the boundary condition parameters:

$$f(\varphi) \sim \begin{cases} 1 + \dfrac{\kappa}{1-4\delta\alpha^2}(\cos\varphi - 2i\delta\alpha\sin\varphi) & \text{for } \dfrac{\kappa}{1 \pm 2\delta\alpha} \ll 1 \\ \dfrac{1}{\sqrt{2}}(1+e^{\mp i\varphi}) + \dfrac{\kappa}{4\sqrt{2}}(e^{\pm i\varphi} + e^{\mp 2i\varphi}) & \text{for } \delta\alpha = \pm\frac{1}{2},\ \kappa \ll 1\ . \end{cases} \tag{14}$$

The mass μ_2 of the χ_2 field is determined by

$$\boxed{\mu_2^2 = \frac{2\pi^2}{L^2}\,\kappa \int d\varphi\, \cos\varphi\, |f(\varphi)|^2} \tag{15}$$

where the normalization $\int d\varphi\, |f|^2 = 1$ is understood. Notice that Eq. (15) must be solved self-consistently. $f(\varphi)$ is determined as a function of κ, but κ in (12) depends on μ_2. In some limiting cases one can find analytic expressions. Suppose that $\delta\alpha = 0$ and $m \ll \mu$ so that $\mu_1 \sim \mu$. Then

$$\mu_2 = \begin{cases} 4\sqrt{2}\,m|\cos\frac{1}{2}\theta|\, e^{-\pi/2\mu L} & \text{for } \mu L \ll 1\ , \\ 4\sqrt{2}\,m|\cos\frac{1}{2}\theta|\left(\dfrac{\mu L e^{\gamma}}{4\pi}\right)^{1/2} & \text{for } \mu L \gg 1 \gg mL(\mu L)^{1/2}|\cos\frac{1}{2}\theta|\ , \\ (4e^{2\gamma}\, m^2|\cos\frac{1}{2}\theta|^2\mu)^{1/3} & \text{for } mL(\mu L)^{1/2}|\cos\frac{1}{2}\theta| \gg 1\ . \end{cases} \tag{16}$$

This contains Coleman's result (2) in the case $mL(\mu L)^{1/2}|\cos\frac{1}{2}\theta| \gg 1$. Similarly the chiral condensate is found to be

$$\langle \overline{\psi}\psi \rangle_\theta = -\frac{\mu_2^2}{4\pi m}\quad . \tag{17}$$

We recognize that there is no puzzle about the fractional power dependence. If a mass m is sufficiently small such that $mL(\mu L)^{1/2}|\cos\frac{1}{2}\theta| \ll 1$, then $\mu_2 \propto m$. A perturbation theory may be employed. An important criterion is not whether m/μ is large or samll, but instead whether the parameter κ is large or small. Depending on the value of κ, the behavior of physical quantities is quite different. This is the content of the statement (3) above.

Furthermore, it is evident that all physical quantities are periodic in θ with a period 2π:

$$\boxed{\text{Periodicity in } \theta \quad = \quad 2\pi}$$

This is so, even though $\frac{1}{2}\theta$-dependence is everywhere. The appearance of the absolute value ensures the 2π periodicity. There appears no indication for the existence of fractional topological number and '4π' periodicity which have been claimed in the literature.[5]

Our analysis can be generalized. One can take into account effects of transitions to higher s states which we have ignored above. We end up with an equation similar to Eq. (11), but this time with one more variable describing dynamics in the s-direction. The equation can be solved numerically.

The $\delta\alpha$ dependence of various physical quantities can also be evaluated numerically. It is found that $|f(\varphi)|^2$ shows significant $\delta\alpha$ dependence for $\kappa < 0.3$, whereas the dependence becomes almost undetectable for $\kappa > 2$. In particular, in the massless fermion limit $mL \to 0$ but with a large L

$$\langle \overline{\psi}\psi \rangle_\theta = \begin{cases} 0 & \text{for } \delta\alpha = 0 \\ -\Big(\dfrac{\mu e^\gamma}{4\pi L} \Big)^{1/2} |\cos \frac{1}{2}\theta| & \text{for } \delta\alpha = \frac{1}{2} \end{cases} \tag{18}$$

In this survey I described a powerful method of analysing the Schwinger model. Pictorially

$$\boxed{\text{Massive } N\text{-flavor Schwinger Model}}$$
$$\Downarrow$$
$$\boxed{N\text{-dim QM}}$$

With this reduction one can solve the model to desired accuracy at least numerically.

There are many intrigueing questions to be answered.

1. For $N = 2$ we have observed that $\langle \overline{\psi}\psi \rangle_\theta$ vanishes at $\theta = \pi$. Does it vanish at $\theta = 2\pi/N$ in general?
2. Is the periodicity in θ always 2π?
3. What happens in the chiral Schwinger model? The above method can be applied with a little modification. Do we have fermion number non-conservation?
4. What happens if each fermion has distinct charge such that the ratio of two charges is irrational? There is no θ vacuum in a rigorous sense, but physics shouldn't depend on the charge parameters so sensitively.

I will come back to these points shortly.

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